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Large-N limit of the "spherical model" of turbulence

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We discuss a "spherical model" of turbulence proposed recently by Mou and Weichman [Phys. Rev. Lett. **70**, 1101 (1993)] and point out its close similarity to the original "random coupling model" of Kraichnan [J. Math. Phys. **2**, 124 (1961)]. The validity of the direct-interaction-approximation (DIA) equations in the limit $N\rightarrow+\infty$ of the spherical model, already proposed by Mou and Weichman, is demonstrated by another method. The argument also gives an alternative derivation of DIA for the random-coupling model. Our proof is entirely nonperturbative and is based on the Martin-Siggia-Rose functional formalism for vertex reversion. Systematic corrections to the DIA equations for the spherical model are developed in a $1/\sqrt{N}$ expansion for a "self-consistent vertex." The coefficients of the expansion are given at each order as the solutions of linear, inhomogeneous functional equations which represent an infinite resummation of terms in the expansion in the bare vertex. We discuss the problem of anomalous scaling in the spherical model, with particular attention given to "spherical shell models" which may be studied numerically.

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I. INTRODUCTION

Kraichnan's "direct interaction approximation," or *DIA equations*, are a classic closure for two-point statistics in turbulence. Originally the (Eulerian) DIA was proposed on the basis of a hypothesis of "maximal randomness" [1]. A few years later, in a very remarkable work, Kraichnan showed that the DIA is the exact solution for $N\rightarrow+\infty$ of a model problem in which N copies of the Navier-Stokes dynamics are coupled together in "collective coordinates" with quenched random interaction phases [2]. (This subject is reviewed in [3].) Recently, Mou and Weichman have proposed an alternative large-N model for the DIA equations, a so-called "spherical model" (SPM) [4]. Although no complete derivation of DIA was given, arguments for its validity were advanced based upon earlier work of Amit and Roginski on a similar large-N model of ϕ^3 field theory with threefold Potts symmetry [5]. Furthermore, Mou and Weichman proposed the spherical model as a suitable basis for calculating short-distance scaling exponents in high Reynolds number turbulence by means of a $1/N$-type expansion based on the Eulerian DIA equations.

In this work we shall confirm the claim of Mou and Weichman, establishing the validity of DIA for the $N\rightarrow+\infty$ limit of the spherical model by a self-consistent argument. Our derivation employs nonperturbative functional techniques developed by Martin, Siggia, and Rose (MSR) [6] and gives also an alternative derivation of DIA for Kraichnan's original "random-coupling model" (RCM). In fact, we point out that the two models, SPM and RCM, are remarkably similar. However, there are important differences between them, and each has technical advantages in certain areas. In particular, it is much easier to calculate systematic corrections (it turns out in $1/N$) to DIA for the SPM. The functional technique used to establish DIA is also applied to derive explicit equations for the corrections. Remarkably, these are *linear equations* at each order, with an inhomogeneous term determined by the previous orders, and, therefore, solvable in principle.

However, we disagree with the program advanced by Mou and Weichman for calculating anomalous exponents in turbulence based upon the Eulerian DIA. In fact, such a program encounters difficulties of a fundamental nature which occur already *at the zeroth order* (i.e., the DIA level) and have been fully analyzed there by Kraichnan [7,8]. To avoid the difficulties would require some sort of Lagrangian formulation of the SPM. We shall discuss this issue carefully in the body of the text. On the other hand, there are some model problems of considerable recent interest, the so-called "shell models" [9–13], in which the above-mentioned difficulties do not occur and the SPM offers a reasonable hope to calculate anomalous scaling. Even in this simplified context, however, we have not yet solved the correction equations.

The contents of this paper are as follows. In Sec. II we describe the spherical model and compare and contrast it with Kraichnan's random-coupling model. We also discuss there the essential difficulties with the proposal of Mou and Weichman to use Eulerian DIA as the basis for calculating anomalous exponents. Section III is devoted to study of the large-N limit problem. We first discuss the standard derivations of DIA for the models, which is based on a "skeleton" expansion of the self-energy involving a "line reversion" of bare propagators for full ones. A great simplification is achieved, however, by performing an additional "vertex reversion," and we review briefly the functional formalism of MSR for this, particularly their exact first-order functional-differential equation for the full vertex as a functional of the bare one. Then we establish the validity of DIA for the large-N limit of the SPM using the functional-differential equation and an asymptotic formula for Wigner $6j$ coefficients heuristically derived by Ponzano and Regge [14]. Thereafter the equations for the corrections in $1/\sqrt{N}$ are derived and discussed. The DIA-level approximation to the effective action is explicitly evaluated and shown to yield one-loop expressions for irreducible functions. Finally, in Sec. IV we discuss the possibilities of calculating anomalous scaling exponents by these methods, particularly for spherical shell models. The DIA equations for the shell models are introduced and outstanding questions stressed, some of which can be addressed numerically.

II. THE "SPHERICAL MODEL"

A. Definition and comparison with the random-coupling model

The Mou-Weichman spherical model [4] is based upon an SU(2) symmetry realized by spin-J multiplets of (complex) velocity fields $\mathbf{v}_\alpha$ labeled by the integer z component $J_z=\alpha$ obeying $|\alpha|\leq J$. Hence there are $N=2J+1$ total components of complex velocities. The velocity fields are subject to the condition $\mathbf{v}_\alpha^*=(-)^{J-\alpha}\mathbf{v}_{-\alpha}$ and J is restricted to even values. The dynamics of the model is taken as

$$\partial_t \mathbf{v}_\alpha+\sum_\beta w_N(\alpha,\beta,\alpha-\beta)P^\perp(\mathbf{v}_\beta\cdot\nabla)\mathbf{v}_{\alpha-\beta}=\nu\Delta\mathbf{v}_\alpha+\mathbf{f}_\alpha \; , \tag{1}$$

where $w_N(\alpha,\beta,\gamma)$ is defined as

$$w_N(\alpha,\beta,\gamma)=\langle (JJ)J_\alpha|(JJ)\beta\gamma\rangle = \sqrt{N}\times(-)^\alpha\begin{pmatrix} J & J & J\\ -\alpha & \beta & \gamma\end{pmatrix} \; , \tag{2}$$

with $\langle (JJ)J\alpha|(JJ)\beta\gamma\rangle$ the Clebsch-Gordan coefficient giving the spin-J component in the irreducible decomposition of the product of two spin-J multiplets, and with $\left(\begin{smallmatrix} J & J & J\\ -\alpha & \beta & \gamma\end{smallmatrix}\right)$ the associated Wigner $3j$ symbol. From the symmetry properties of the Wigner $3j$ symbols, discussed, for example, in [15], follow the basic properties of the coupling coefficients w_N:

$$w_N(\alpha,\beta,\gamma)=0 \quad \text{if } \alpha\neq\beta+\gamma \; , \tag{3}$$

$$w_N(\alpha,\beta,\gamma)=w_N(\alpha,\gamma,\beta) \; , \tag{4}$$

$$w_N(\alpha,\beta,\gamma)^*=w_N(\alpha,\beta,\gamma) \; , \tag{5}$$

$$w_N(\alpha,\beta,\gamma)=(-)^{\alpha+\gamma}w_N(-\gamma,\beta,-\alpha) \; . \tag{6}$$

Furthermore, for any element $U\in\text{SU}(2)$, it follows that

$$D^J_{\alpha\alpha'}(U^{-1})D^J_{\beta\beta'}(U)D^J_{\gamma\gamma'}(U)w_N(\alpha',\beta',\gamma')=w_N(\alpha,\beta,\gamma) \; , \tag{7}$$

where D^J are the usual Wigner D^J matrices [15]. As a consequence of this property, the SPM dynamics is covariant to general SU(2) transformations of the fields:

$$\mathbf{v}'_\alpha=D^J_{\alpha\beta}(U)\mathbf{v}_\beta \; . \tag{8}$$

From the second relation on the w_N's it can be shown that the Liouville theorem holds (by an argument exactly analogous to that made for the RCM in [3]). Furthermore, the fourth relation on the w_N's implies that the following "energy function":

$$E(t)=\tfrac{1}{2}\sum_{\alpha=-J}^{J}\int d^d\mathbf{r}|\mathbf{v}_\alpha(\mathbf{r},t)|^2 \tag{9}$$

is conserved by the SPM dynamics. Except for the case $J=0$, which is the usual Navier-Stokes dynamics, there is no analog of the Galilei transformations under which Navier-Stokes equations are covariant. If it were true that $w_N(\alpha,0,\gamma)=w_N\delta_{\alpha,\gamma}$, with w_N a function of $N=2J+1$ *only* (i.e., independent of α,γ), then it would be possible to define such a set of transformations, as

$$\mathbf{v}'_\alpha(\mathbf{r},t)=\mathbf{v}_\alpha(\mathbf{r}-\mathbf{u}t,t) \tag{10}$$

for $\alpha\neq0$ and

$$\mathbf{v}'_0(\mathbf{r},t)=\mathbf{v}_0(\mathbf{r}-\mathbf{u}t,t)+\frac{\mathbf{u}}{w_N} \; . \tag{11}$$

In fact, it is easy to see by checking the tables of Clebsch-Gordan coefficients that the required property is untrue and there is thus no true Galilei symmetry for the SPM.

This model is actually quite similar to Kraichnan's random-coupling model in its original 1961 formulation [2]. In that model there were the same number of variables with the same labels as above, except that $\mathbf{v}_\alpha^*=\mathbf{v}_{-\alpha}$ without the sign factor. Furthermore, the dynamics took also the sum form but instead with

$$w_N(\alpha,\beta,\gamma)=\frac{1}{\sqrt{N}}\phi_{\alpha,\beta,\gamma} \; , \tag{12}$$

where $\phi_{\alpha,\beta,\gamma}$ was chosen as a *completely random phase* over the indices α,β,γ subject only to the restrictions

$$\phi_{\alpha,\beta,\gamma}=0 \quad \text{if } \alpha\neq\beta+\gamma \; , \tag{13}$$

$$\phi_{\alpha,\beta,\gamma}=\phi_{\alpha,\gamma,\beta} \; , \tag{14}$$

$$\phi^*_{\alpha,\beta,\gamma}=\phi_{-\alpha,-\beta,-\gamma} \; , \tag{15}$$

$$\phi_{\alpha,\beta,\gamma}=\phi_{-\gamma,\beta,-\alpha} \; . \tag{16}$$

The first restriction guaranteed that the model had an exact Z(N) symmetry corresponding to the set of transformations $\mathbf{v}'_\alpha=e^{2\pi i\alpha n/N}\mathbf{v}_\alpha$ with $n\in Z(N)$. Notice that this symmetry is included as a subgroup in the SU(2) symmetry of the SPM, associated with rotations about the z axis in spin space by integer multiples of the angle $2\pi/N$. Like the SPM, the RCM enjoys a Liouville theorem and conservation of the energy Eq. (9). An important difference exists, however, with respect to Galilei covariance of the RCM and SPM. It turns out that $\phi_{\alpha\beta\gamma}=1$ may be imposed in the RCM as an additional restriction whenever $\alpha\beta\gamma=0$, without affecting the validity of DIA in the limit. In that case the model is Galilei covariant as in Eqs. (10) and (11) with $w_N=1/\sqrt{N}$. [However, it has been pointed out to us by P. Weichman that it is possible to consider a spherical model based on U(N) symmetry, with an extra "zero-spin" generator, giving a Galilei-like symmetry.] The resemblance of the SPM to the RCM is made even more striking by noting a (semiclassical) large-J asymptotic formula for the $3j$ symbols which has been heuristically derived by Ponzano and Regge [see Eq.

(2.6) in [14]], from which it follows in the SPM that

$$w_N(\alpha,\beta,\gamma)\sim\frac{1}{\sqrt{Nf(k,p,q)}}\cos[Ng(k,p,q)] \quad (17)$$

for $N\rightarrow+\infty$ with f,g fixed smooth functions of the variables $k=\alpha/J$, $p=\beta/J$, and $q=\gamma/J$. Hence the coupling of the velocity spin components $\mathbf{v}_\alpha$ for the SPM has the same strength $\sim 1/\sqrt{N}$ as in the RCM and a sign which, while not totally random, is very rapidly oscillating over the values of α,β,γ as $N\rightarrow+\infty$.

The two models may indeed be analyzed by essentially the same methods in the limit $N\rightarrow+\infty$, as we discuss in Sec. III. Each model has some particular advantages and disadvantages. Because the RCM is Galilei covariant, it may be used to prove realizability of DIA equations for inhomogeneous situations where the mean flow does not vanish [2]. It seems to be a somewhat more flexible method than the SPM. On the other hand, the appearance of the quenched randomness in the interaction phases of the RCM is probably a liability for numerical simulation of the model. Here the deterministic character of the SPM is an advantage. Furthermore, a feature of the RCM which greatly complicates the analysis of large-N corrections to the DIA is that factorization of averages of correlation functions over the random phases,

$$\overline{G_p(1\cdots p)G_q(1'\cdots q')} = \overline{G}_p(1\cdots p)\overline{G}_q(1'\cdots q')+O\left[\frac{1}{N}\right], \quad (18)$$

holds only up to corrections of order $1/N$. In the SPM, of course, this problem does not occur, and, as we see, the corrections may be obtained in a fairly explicit form.

B. The problem with Eulerian DIA

However, we strongly disagree with the program put forth by Mou and Weichman in [4] to calculate high Reynolds number scaling exponents—in particular the energy spectral exponent, which they denote as ζ—by an expansion around the solution of Eulerian DIA. To clarify the disagreement, we quote them verbatim [4]: "The idea we propose is that the DIA equations (8) represent an exact solution in a special limit, which is *continuously* related, via N, to the real turbulence problem. The equations should thus be taken *at face value*. Previous work which has concentrated on modifying them to obtain the $\frac{5}{3}$ law thus appears misguided. We view the $\frac{3}{2}$ law as an amazingly accurate zeroth-order approximation in a systematic approximation for $\zeta(N)$."

This point of view seems to us certainly wrong. In fact, the difficulty with the Eulerian DIA, as discussed by Kraichnan in [7,8] is *not* that it fails to produce $\frac{5}{3}$ law. Instead the problem is that Eulerian DIA violates basic physical principles and it is this violation which is, in turn, responsible for the spurious $\zeta=\frac{3}{2}$ law. The precise point is that Galilei covariance of the Navier-Stokes dynamics implies that from any ensemble of solutions a new ensemble of solutions may be produced by performing independent random Galilei transformations on the realizations of the original ensemble. It is easy to check that for an initial homogeneous ensemble the single-time Eulerian velocity cumulants (as well as multitime Lagrangian cumulants) in the new ensemble differ from those in the original ensemble only by the addition of the corresponding cumulant of the random boost velocity:

$$\langle\langle v(1)\cdots v(p)\rangle\rangle^C=\langle v(1)\cdots v(p)\rangle^C+\overline{u^p}^C , \quad (19)$$

where $\langle\langle\ \rangle\rangle^C, \langle\ \rangle^C$ denote cumulant averages in the new and original ensembles, respectively, and $\overline{u^p}^C$ is the pth-order cumulant of the random boost velocity. This addition affects the original velocity cumulants *at zero wave number only,* so that the transformation law Eq. (19) is referred to as "random Galilei invariance." Notice, in particular, that equal-time velocity triple moments, which determine the instantaneous rate of subscale energy transfer to small length scales, must be invariant (away from zero wave number). However, the Eulerian DIA expression for the triple moment is *not* invariant to random Galilei transformations, because it is a closure in terms of Eulerian two-time moments, which are not invariant. This noninvariance is a violation of fundamental principles. It follows directly from this defect that the rate of energy transfer in Eulerian DIA is set by an overall convection time scale, determined from the rms velocity of the large scales, rather than by an intrinsic, local eddy-turnover time. It is this fact that leads to the $\frac{3}{2}$ energy law rather than Kolmogorov's $\frac{5}{3}$ law. On the other hand, one may consider instead of Navier-Stokes dynamics a system like Kraichnan's "modified Navier-Stokes system" [7], in which all wave number triad interactions are removed such that the ratio of maximum to minimum wave number exceeds some threshold value and which gives a crude representation of a transformation to "quasi-Lagrangian" coordinates. Then the DIA equations give a transfer rate which is determined by a local turnover time and the $\frac{5}{3}$ law results. (There are some difficult points in the above argument, which we will not discuss here.)

There is another, but related, way to make the argument against the Mou-Weichman proposal, based on renormalization group (RG) ideas. If one sets up a natural RG for fully developed turbulence, then one finds that the *fixed point* contains as its only dimensional parameter the mean dissipation $\bar{\epsilon}$. Anomalous exponents associated with intermittency corrections to Kolmogorov power laws enter in as corrections to scaling associated to the integral scale L. The form of the corrections may be determined by the operator-product expansion, e.g., for longitudinal structure functions, as

$$\langle\{\hat{l}\cdot[\mathbf{v}(\mathbf{r}+\mathbf{l})-\mathbf{v}(\mathbf{r})]\}^p\rangle\sim(\bar{\epsilon}l)^{p/3}\left[\frac{L}{l}\right]^{-x_p} , \quad (20)$$

where $x_p<0$ is an "anomalous exponent" (see [16,13]). Now, suppose one wants to try to calculate the anomalous exponents x_p under the assumption that they have an asymptotic expansion in $1/N$ of the form

$$x_p=\sum_{k\geq 1}\frac{c_k^{(p)}}{N^k} . \quad (21)$$

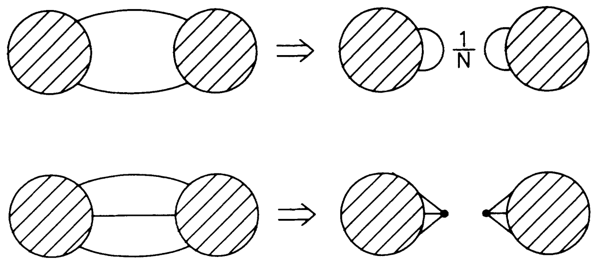

FIG. 1. Factorization of two- and three-particle reducible symbols.

Then, it is clear from the above expression in Eq. (20) that the anomalous exponents will appear in a usual form as logarithmic divergences in the $1/N$ expansion *but only if one expands around an* $N=+\infty$ *limit which has Kolmogorov scaling*. If, on the other hand, one expands around the solution of the Eulerian DIA, then additional infrared divergences will appear which come just from expanding around the "wrong limit." These additional divergences will only reflect convective "sweeping" and shall be very difficult to disentangle from any divergences which might be due to physical intermittency.

Therefore, we do not believe there is any hope of success for the Mou-Weichman program applied to the Navier-Stokes dynamics in its Eulerian form. A suitable Lagrangian formulation of the spherical model would have to be employed, which remains to be devised. On the other hand, the tenability of the basic concept can be tested on simple models like the "modified Navier-Stokes system" [7] or the "shell models" [9–13], which lack the previous difficulties. Since these models seem to have the same types of corrections to Kolmogorov scaling as the Navier-Stokes system and the simple DIA leads in them to the Kolmogorov scaling laws, the spherical model might provide a suitable basis for calculating the scaling corrections. On the other hand, the failure of the idea there would obviate the need to work out its (difficult) Lagrangian extension to the Navier-Stokes system. These matters are discussed further in Sec. III. We only remark here that a simple $1/N$ expansion of the traditional type cannot be expected to hold for the SPM, because in addition to factors of $1/\sqrt{N}$ which appear in the interaction, there is also the rapidly oscillating part. We turn now to the study of the $N\rightarrow+\infty$ limit itself.

III. LARGE-N LIMIT

A. Arguments using line-reverted expansions

The traditional argument for the DIA is based upon the skeleton or irreducible expansion of the self-energy, going back to Kraichnan's original derivation for the RCM [2] (see also [3]). In this method the self-energy of the model (formulated as a formal field-theory problem) is expanded as an infinite series in the bare vertex and the

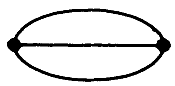

FIG. 2. Graph for the $3j$ symbol.

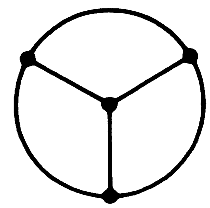

FIG. 3. Graph for the $6j$ symbol.

full propagators. The coefficients in this expansion for the RCM, Kraichnan's "irreducible cycles," are found all to vanish in the limit $N\rightarrow+\infty$ except for the lowest-order term, corresponding to DIA, which has coefficient 1. This gives a formal derivation of the validity of DIA for the RCM, although some delicate questions concerning the convergence of the skeleton expansion and the commutation of limit operations are begged.

The argument of Amit and Roginsky for validity of the DIA equation in their large-N model of ϕ^3 field theory [5] is essentially the same as Kraichnan's. The coefficients of the skeleton expansion are now certain *Wigner* $3nj$ *symbols* [15]. These latter are exactly represented by the $(n+1)$-loop vacuum graphs in a theory whose vertices are given by Wigner $3j$ symbols and whose propagator legs are given by Kronecker deltas $(-)^{j-m}\delta_{j,j'}\delta_{m,-m'}$. The conventional $3nj$ symbols are the three-particle irreducible ones, in terms of which all others may be written (but note that all the two-particle irreducible symbols appear in the skeleton expansion of the SPM self-energy). Indeed, a remarkable feature of the $3nj$ symbols is that they have exact *factorization properties* for the two- and three-particle reducible symbols, as shown in Fig. 1. Similar factorization properties hold for the ϕ factors or "cycles" in Kraichnan's RCM only in the limit $N\rightarrow+\infty$. The lowest order symbol, with $n=1$, which is represented by the graph in Fig. 2, has the value 1. It corresponds to the DIA terms in the self-energy. The higher-order $3nj$ symbols are expected to decay like some powers of $1/\sqrt{N}$. For example, the $6j$ symbol $\{{J\,J\,J \atop J\,J\,J}\}$, which corresponds to the graph in Fig. 3, decays as $N^{-3/2}$ according to the semiclassical formula of Ponzano and Regge [14]. This formula may be stated more precisely in terms of a geometric representation of the $6j$ symbol $\{{j_1\,j_2\,j_3 \atop l_1\,l_2\,l_3}\}$, by a tetrahedron with sides of length $j_1+\frac{1}{2},j_2+\frac{1}{2},j_3+\frac{1}{2},l_1+\frac{1}{2},l_2+\frac{1}{2},l_3+\frac{1}{2}$, as shown in Fig. 4. If the faces are indexed by $h=1,2,3,4$, the length of the edge at the intersection of sides h,k is denoted j_{hk}, and the angle between outward normals of the faces is denoted as θ_{hk}, then the large-J asymptotic (semiclassical) formula pro-

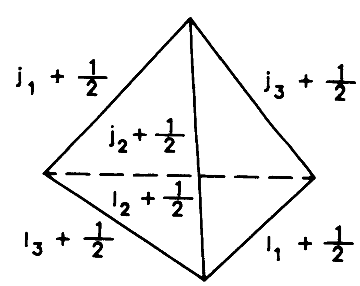

FIG. 4. Ponzano-Regge tetrahedral representation of the $6j$ symbol.

posed by Ponzano and Regge was

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} \cos\left[\sum_{h<k} j_{hk}\theta_{hk} + \frac{\pi}{4} \right] . \quad (22)$$

The V appearing in the denominator is the (signed) volume of the tetrahedron and the stated formula only applies for $V > 0$. The expression is clearly reminiscent of a WKB-type formula, and formulas also for $V < 0$ and for the transitional region $V \approx 0$ were worked out by a connection procedure. None of the formulas are rigorously derived, but they satisfy all known symmetry properties of the $6j$ symbols. Furthermore, they asymptotically satisfy all known sum relations, certain subsets of which are known to characterize the $6j$ coefficients up to a phase. The asymptotic expressions give amazingly accurate values even for rather small J [14]. Therefore, the expressions seem to be quite plausible, if not rigorously proved. From the Ponzano-Regge formulas for the $6j$ coefficient, upper bounds can be established on certain other classes of $3nj$ symbols. The results are indicative that all of the coefficients in the skeleton expansion of the self-energy for the SPM, besides the DIA term, are vanishing in the limit $N \to +\infty$. However, it does not seem to be presently possible to show this directly by study of the $3nj$ symbols.

It turns out there is an argument for validity of DIA which avoids the difficult study of infinitely many $3nj$ coefficients and is based on the Ponzano-Regge formula for the $6j$ coefficient alone. The approach is entirely nonperturbative, employing the MSR functional formalism for vertex reversion [6]. We shall very briefly review this formalism here and refer to the original works for full details.

B. The functional formalism of vertex reversion

The famous paper of Martin, Siggia, and Rose [6] achieved a field-theory formulation of classical stochastic dynamics which systematized the previous perturbation approaches of Wyld [17], Kraichnan [2], and Edwards [18] (although the MSR method itself is nonperturbative). In this approach a "doublet" field was used

$$\Phi = \begin{bmatrix} \mathbf{v} \\ i\hat{\mathbf{v}} \end{bmatrix} , \quad (23)$$

in which $\hat{\mathbf{v}}$ is a "response field" whose correlators with the usual velocity field, $i^p\langle \mathbf{v}(0)\hat{\mathbf{v}}(1)\cdots\hat{\mathbf{v}}(p)\rangle$, are higher-order (nonlinear) response functions. By formally exact procedures, applicable to any canonical field theory, MSR derived the following set of *Schwinger-Dyson integral equations* for the self-energy Σ and full vertex (or irreducible three-point function) Γ_3:

$$G_2^{-1} = [G_2^{(0)}]^{-1} - \gamma_3 G_1 - \Sigma \quad (24)$$

and

$$\Sigma = \tfrac{1}{2}\gamma_3 G_2 G_2 \Gamma_3 . \quad (25)$$

In these equations and elsewhere, G_p represents a pth-order cumulant of Φ. As is well known, the DIA equations are the approximation to Eqs. (24) and (25) "ignoring vertex corrections," i.e., with γ_3 replacing Γ_3 in Eq. (25). Another equation follows from the fact that $\Gamma_3 = \delta\Gamma_2/\delta G_1$, where $\Gamma_2 = \gamma_3 G_1 + \Sigma(G_2)$ is the irreducible two-point function, and the fact that Σ depends upon G_1 really only implicitly through G_2. The extra equation is

$$\Gamma_3 = \gamma_3 + (\delta\Sigma/\delta G_2) G_2 G_2 \Gamma_3 . \quad (26)$$

This equation along with Eq. (25) can be used iteratively to generate the skeleton expansions of Σ and Γ_3.

For many purposes, however, such an expansion in the full propagator but the bare vertex is inadequate. In fact, as we argue elsewhere, any finite-order expansion in terms of the bare vertex is inadequate to calculate intermittency corrections to Kolmogorov scaling and some sort of "infinite vertex resummation" is required [19]. It is possible to develop an expansion for γ_3 in terms of Γ_3 by a formal reversion of the skeleton expansion of Γ_3 and, then substituting the expression for γ_3 everywhere the latter appears, to elaborate a perturbation representation of all statistics in terms of Γ_3. In fact, however, the reversion procedure is a nonperturbative operation. The theory of the reversion is greatly simplified by the observation that there is a duality between bare and full functions, since the full pth-order moment $M_p(1,\ldots,p) \equiv \langle\Phi(1)\cdots\Phi(p)\rangle$ is obtained as the functional derivative

$$M_p(1,\ldots,p) = p!\frac{\delta W}{\delta\gamma_p(1,\ldots,p)} \quad (27)$$

of the generating functional

$$W = \ln\int \mathcal{D}\Phi \exp(S[\Phi]) , \quad (28)$$

where

$$S[\Phi] = \tfrac{1}{2}\Phi(1) i\sigma^{(2)}\partial_{t_1}\Phi(1) + \sum_{k\geq 1}\frac{1}{k!}(1,\ldots,k)\Phi(1)\cdots\Phi(k) \quad (29)$$

is the field-theory action defined in terms of bare interaction vertices γ_k. This observation allows the reversion between the γ's and the M's to be accomplished by Legendre transformation of W [20]. Up to the third order, this Legendre transform is defined as

$$I^{(3)}(G_1, M_2, M_3) \equiv W - \gamma(1)G_1(1) - \frac{1}{2!}\gamma_2(12)M_2(12) - \frac{1}{3!}\gamma_3(123)M_3(123) . \quad (30)$$

Note that

$$\frac{\delta I^{(3)}}{\delta M_k(1\cdots k)} = -\frac{1}{k!}\gamma_k(1\cdots k) \quad (31)$$

for $k \leq 3$ whereas

$$\frac{\delta I^{(3)}}{\delta\gamma_k(1\cdots k)} = \frac{1}{k!}M_k(1\cdots k) \quad (32)$$

for $k > 3$. The same methods can obviously be used for higher and lower order functions. Since the relations hold that $M_2 = G_2 + G_1 G_1$, $M_3 = G_3 + 3G_2 G_1 + G_1 G_1 G_1$, and $G_3=\Gamma_3G_1G_1G_1$ the first equation (31) formally accomplishes the reversion between γ_3 and Γ_3. In fact, the "three-body effective action" $I^{(3)}$ may be shown to be of the form

$$I^{(3)}(G_1,G_2,\Gamma_3) =\tfrac{1}{2}\mathrm{Tr}\{i\sigma^{(2)}\partial_{t_1}[G_2(11')+G_1(1)G_1(1')]+\ln G_2\} -\frac{1}{2\times 3!}\Gamma_3G_2G_2G_2\Gamma_3+\Omega^{(3)}(G_2,\Gamma_3)+\mathrm{const}\ . \quad (33)$$

The functional $\Omega^{(3)}$ appearing here can be characterized diagrammatically as the sum of all three-particle irreducible vacuum graphs in a theory with propagator legs G_2 and vertices Γ_3 [21] (although it is defined above by the Legendre transform without any reference to perturbation theory). This latter quantity is really a functional only of the dimensionless "truncated vertex" $\overline{\Gamma}_3\equiv\Gamma_3G_2^{1/2}G_2^{1/2}G_2^{1/2}$ to which is attached one-half of the propagator legs (see [6,20]). If one makes the inverse Legendre transform

$$W\equiv I^{(3)}+\gamma(1)G_1(1)+\frac{1}{2!}\gamma_2(12)M_2(12) +\frac{1}{3!}\gamma_3(123)M_3(123)\ , \quad (34)$$

then the stationarity of W under variations of $\overline{\Gamma}_3$ gives the equation

$$\overline{\Gamma}_3(123)=\overline{\gamma}_3(123)+\overline{K}_3(123;\overline{\Gamma}_3)\ , \quad (35)$$

in which $\overline{\gamma}_3$ is the "truncated" bare vertex $\gamma_3G_2^{1/2}G_2^{1/2}G_2^{1/2}$ and

$$\overline{K}_3(123;\overline{\Gamma}_3)=3!\frac{\delta\Omega^{(3)}}{\delta\overline{\Gamma}_3(123)}\ . \quad (36)$$

Therefore, diagrammatically $\overline{K}_3$ is the sum of all three-particle irreducible three-point graphs with vertices given by $\overline{\Gamma}_3$ (and propagator lines 1) [21].

For our purposes there is an even more useful result of MSR, which is an exact first-order functional-differential equation for the full "truncated" vertex $\overline{\Gamma}_3$ as a functional of the bare one $\overline{\gamma}_3$. This equation is easily derived by substituting the Eq. (25) for Σ into Eq. (26) and then noting that $\overline{\Gamma}_3$ depends upon G_2 only implicitly through its dependence upon $\overline{\gamma}_3$. The equation takes the explicit form

$$\overline{\Gamma}_3=\overline{\gamma}_3+\tfrac{1}{4}\overline{\gamma}_3\overline{\Gamma}_3^2+\tfrac{3}{4}\overline{\gamma}_3^2\left[\frac{\delta\overline{\Gamma}_3}{\delta\overline{\gamma}_3}\right]\overline{\Gamma}_3\ . \quad (37)$$

The equation may be solved by iteration to yield $\overline{\Gamma}_3$ as a functional power series in $\overline{\gamma}_3$. Diagrammatically the equation may be represented as shown in Fig. 5. The more compact expression in Eq. (37) does not distinguish between different contributions of the same order of terms with different structure, and it is difficult to do so without a confusing crowd of indices. The diagrammatic representation is often most transparent and we shall use it for convenience in our arguments.

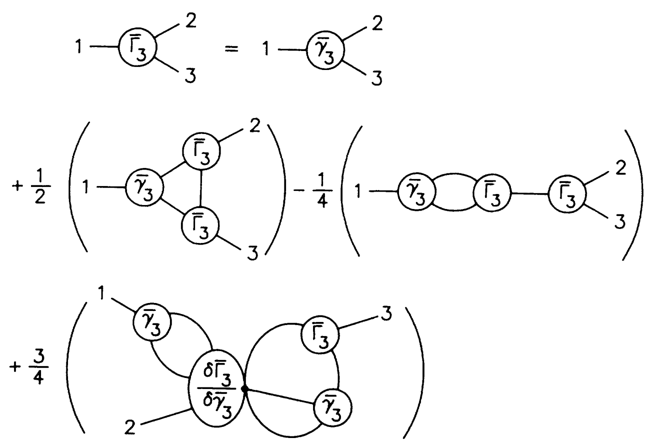

FIG. 5. MSR functional differential equation for the full vertex.

C. Application to the large-N behavior of the spherical model

All of the previous formalism applies to the spherical model just as well as to the original Navier-Stokes system. An important fact which greatly simplifies the analysis is that the full propagator G_2 and vertex Γ_3 in the SPM have a very simple dependence upon the spin-projection indices. In fact,

$$G_2(\alpha 1,\beta 2)=\delta_{\alpha,\beta}G_2^{(N)}(12) \quad (38)$$

and

$$\Gamma_3(\alpha 1,\beta 2,\gamma 3)=\sqrt{N}\times(-)^{\alpha}\begin{bmatrix} J & J & J\\ -\alpha & \beta & \gamma\end{bmatrix}\Gamma_3^{(N)}(123)\ , \quad (39)$$

where numbers $1,2,3,\ldots$ indicate all other variables besides spin projections. [The factor $\sqrt{N}$ is included in Eq. (39) so that $\Gamma_3^{(N)}=O(1)$.] These facts are due directly to the SU(2) transformation properties of the functions: see [15]. The function $\Gamma_3^{(N)}$ can be obtained by "contracting" the full vertex with a Wigner $3j$ symbol:

$$\Gamma_3^{(N)}(123)=\frac{1}{\sqrt{N}}\sum_{\alpha,\beta,\gamma}(-)^{\alpha}\begin{bmatrix} J & J & J\\ -\alpha & \beta & \gamma\end{bmatrix}\Gamma_3(\alpha 1,\beta 2,\gamma 3)\ . \quad (40)$$

In applying the functional-differential equation Eq. (37) to the SPM, however, it is crucial to note that it is an equation for $\overline{\Gamma}_3(\alpha 1,\beta 2,\gamma 3)$ and *not* for $\overline{\Gamma}_3^{(N)}(123)$. In particular, the functional derivative which appears in that equation must be defined so that

$$\frac{\delta\overline{\gamma}_3(\alpha 1,\beta 2,\gamma 3)}{\delta\overline{\gamma}_3(\alpha' 1',\beta' 2',\gamma' 3')}=\delta(\alpha 1,\beta 2,\gamma 3;\alpha' 1',\beta' 2',\gamma' 3')\ . \quad (41)$$

The right-hand side of this equation is the "totally symmetric δ function," which is given by $\delta(123;1'2'3')=\mathcal{S}_{123}\delta(11')\delta(22')\delta(33')$ with $\mathcal{S}_{123}$ a symmetrizer in its indices.

It is easiest to give the argument for validity of DIA in terms of an equation for $\overline{K}_3=\overline{\Gamma}_3-\overline{\gamma}_3$. By substituting

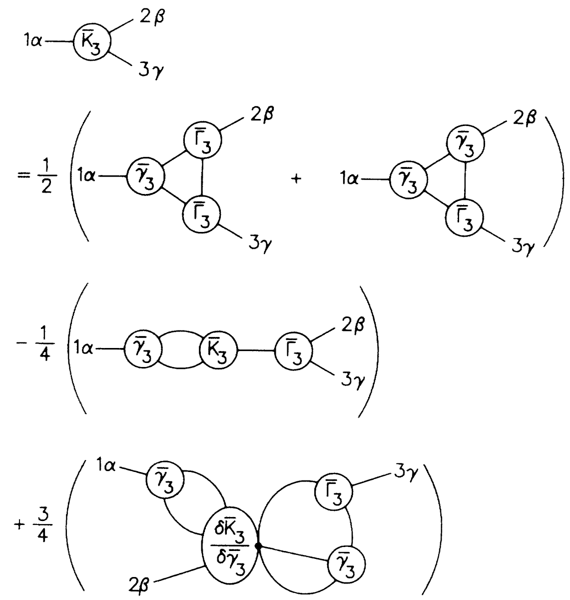

FIG. 6. Functional differential equation for the vertex correction.

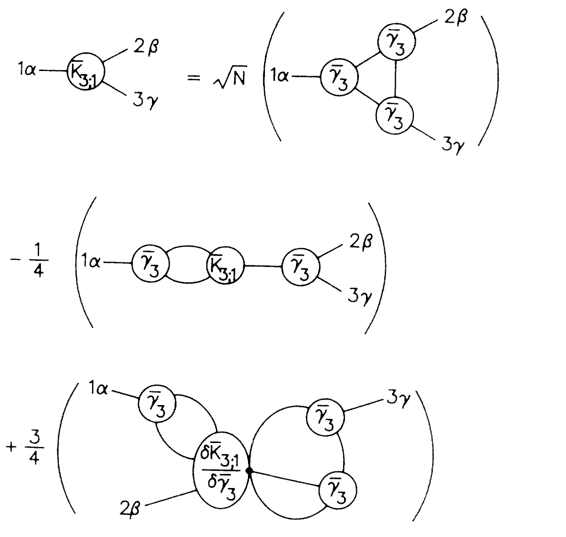

FIG. 7. Linearized equation for the lowest order vertex correction.

$\overline{\Gamma}_3=\overline{\gamma}_3+\overline{K}_3$ in Eq. (37), it is straightforward to derive the equation shown in Fig. 6. All of the terms in this equation are proportional to the Wigner $3j$ symbol $\begin{pmatrix} J & J & J \\ -\alpha & \beta & \gamma \end{pmatrix}$, which may be factored out along with the overall $\sqrt{N}$ factor associated to the vertex function $\overline{K}_3$ on the left-hand side. Notice that the first two terms on the right-hand side are then proportional to the Wigner $6j$ symbol $\begin{Bmatrix} J & J & J \\ J & J & J \end{Bmatrix}$ times a factor $N=N^{3/2}N^{-1/2}$, which comes from the overall $\sqrt{N}$'s in front of the vertex functions. The total coefficient, which is $O(1/\sqrt{N}\,)$, multiplies a term which now depends just upon the functions $\overline{\gamma}_3^{(N)}(123)$ and $\overline{\Gamma}_3^{(N)}(123)$. The next term on the right-hand side is a "DIA-like" term proportional to 1, as may be seen by using the two-particle factorization property to give a factor $1/N$, which exactly cancels the factor N from the overall $\sqrt{N}$'s. The final term has a much more complicated N dependence through all possible $3nj$ coefficients. However, it is not necessary to elucidate this to verify that $\overline{K}_3=0$, i.e., $\overline{\Gamma}_3=\overline{\gamma}_3$, satisfies this equation to leading order. Indeed, substituting $\overline{K}_3=0$, one finds that the equation is satisfied to terms of order $O(1/\sqrt{N}\,)$ (coming from the first two terms on the right-hand side). This provides a self-consistent proof of validity of DIA, in the sense that it is consistent with the exact Eq. (37) to assume a solution of the form $\overline{\Gamma}_3=\overline{\gamma}_3[1+O(1/\sqrt{N}\,)]$. If a solution of the functional-differential equation can be constructed in the form of a series,

$$\overline{\Gamma}_3=\overline{\gamma}_3+\sum_{k\geq 1}\frac{\overline{K}_{3;k}}{N^{k/2}}\ , \qquad (42)$$

satisfying $\overline{\Gamma}_3(\overline{\gamma}_3=0)=0$, and if the Eq. (37) has a *unique* solution subject to that condition, then this series must represent the full vertex of the statistical problem. (The latter is guaranteed to be one such solution.)

Equations for the corrections terms to DIA in the assumed form may be obtained by substituting the expansion for $\overline{K}_3$ into its functional-differential equation and keeping only the leading nonvanishing terms. One finds the *linear* equation for $\overline{K}_{3;1}$ (shown in Fig. 7). Essentially this equation was obtained by linearizing the full differential equation around the zeroth-order solution $\overline{K}_3=0$, or $\overline{\Gamma}_3=\overline{\gamma}_3$, with an inhomogeneity determined from the solution of the zeroth order (DIA). Notice it follows from the order of its inhomogeneity, $O(1/\sqrt{N}\,)$, that it would not be consistent with uniqueness to have a slower decay of the first-order correction, e.g., $\overline{K}_{3;1}=O(1/N^{\alpha})$, $\alpha<\frac{1}{2}$. Indeed, the linear equation for $\overline{K}_{3;1}$ would then be *homogeneous* and therefore have immediately $\overline{K}_{3;1}\equiv 0$ as one solution. However, if $\overline{K}_{3;1}\neq 0$ were the leading-order correction, as assumed, then this linear equation would have at least *two* distinct solutions, either of which could be used as the basis of an interative construction of a solution of the full equation perturbatively in $1/\sqrt{N}$—violating the assumption of uniqueness. Therefore, if this assumption is correct, no $3nj$ symbol may decay slower than $O(1/\sqrt{N}\,)$, or else the summed contribution to $\overline{\Gamma}_3$ from such symbols must vanish identically. On the other hand, the solution of the linear equation for $\overline{K}_{3;1}$ will contain not only terms $O(1/\sqrt{N}\,)$, but also higher order terms as well. The sense in which it is the "first correction" is that it is guaranteed to give *all* of the $O(1/\sqrt{N}\,)$ contributions. The procedure may be continued to higher orders. At each stage one has a linear equation (with structurally the same linear operator at each stage) and with an inhomogeneity determined by the solutions of the previous stages. The solution at the kth stage will give $\overline{\Gamma}_3^{(N)}$ correct up to terms of order $O(N^{-k/2})$. The propagator G_2 which is correct to order $O(N^{-k/2})$ may be obtained by solving the Schwinger-Dyson equation Eq. (25) with the vertex function Γ_3 to that order.

The remarkable simplifying feature of the equation Eq. (37) is that it reduces the whole proof of the DIA to the asymptotic decay $O(N^{-3/2})$ of the $6j$ coefficient, which follows from the Ponzano-Regge formula. Furthermore,

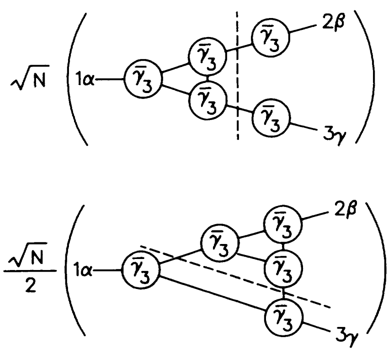

FIG. 8. Two reducible vertex contributions in the iterative solution.

the same proof works for Kraichnan's random-coupling model and is completely nonperturbative. [However, for the RCM the factorization property Eq. (18) is also required and the only proof we know of that still refers to the perturbation expansion.]

An important point is that the solution of the linear equation for $\overline{K}_{3;1}$ sums an infinite class of diagrams in the expansion of $\overline{\Gamma}_3$ in terms of $\overline{\gamma}_3$. In fact, solving the linear equation for $\overline{K}_{3;1}$ by iteration generates an infinite set of terms. After the $O(1)$ term proportional to the $6j$ symbol which appears explicitly, the first few terms generated in the iteration process (shown in Fig. 8) are actually $O(1/\sqrt{N})$ (using the three-particle factorization property, their spin parts are just a product of two $6j$ coefficients) and another one (shown in Fig. 9) is proportional to a $9j$ coefficient and therefore possibly an $O(1)$ contribution to $\overline{K}_{3;1}$ (see Appendix I in [5]). The appearance of infinitely many terms in the solutions is very important, since only approximations to Γ_3 involving an "infinite vertex resummation" can succeed to calculate anomalous scaling exponents [19]. (However, we have not shown that there are infinitely many contributions to $\overline{\Gamma}_3^{(N)}$, which are $\sim 1/\sqrt{N}$ in this solution. The calculations of Amit and Roginsky in Appendix I of [5] give an indication, on the other hand, that there may be infinitely many terms contributing $\sim 1/N$ in the next stage.)

Our method here is not actually a perturbation approach in the sense of Feynman-type diagrams, although we make an expansion in the small parameter $1/\sqrt{N}$. The idea instead is to generate a sequence of successive approximations to $\overline{\Gamma}_3$. In principle, all statistics may be obtained from a knowledge of the full vertex $\overline{\Gamma}_3$ as a functional of the bare one $\overline{\gamma}_3$ to a given order. In fact, knowing the functional $\overline{K}_3$ to a given order gives the function

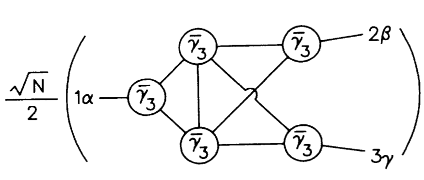

FIG. 9. An irreducible ($9j$) contribution in the iterative solution.

$\Omega^{(3)}(\overline{\Gamma}_3)$ to the same order by a "line integral" in function space

$$\Omega^{(3)}(\overline{\Gamma}_3)=\frac{1}{3!}\int_0^{\overline{\Gamma}_3} d\overline{\Gamma}_3'(123)\overline{K}_3(123;\overline{\Gamma}_3') , \tag{43}$$

and this gives an approximation to the three-body effective action $I^{(3)}$ when substituted into Eq. (33). From this functional all correlations can be obtained. For example, making a twofold inverse Legendre transform

$$I^{(1)}(G_1)=I^{(3)}(G_1,G_2,\Gamma_3)+\frac{1}{2!}\gamma_2(12)M_2(12) + \frac{1}{3!}\gamma_3(123)M_3(123) \tag{44}$$

gives the usual one-body effective action $I^{(1)}$, which is the generating functional for the irreducible functions Γ_p by taking p functional derivatives with respect to G_1.

D. Evaluation of the effective action and irreducible functions at the DIA level

For an explicit evaluation of the effective action $I^{(1)}$ to a given order in $1/\sqrt{N}$, however, the above procedure is not the most convenient. For one thing, the iterative solution for $\overline{K}_3$ gives it a function of $\overline{\gamma}_3$ rather than of $\overline{\Gamma}_3$ as required above. This problem can be avoided and one Legendre transform step eliminated by observing that the expansion of $\overline{\Gamma}_3$ in powers of $1/\sqrt{N}$ also gives immediately the expansion of the self-energy Σ through the Schwinger-Dyson equation Eq. (25). Furthermore, the two-body effective action $I^{(2)}$ can be written directly in terms of the self-energy Σ. In fact, since $-\frac{1}{2}\gamma_2(12)=\delta I^{(2)}/\delta G_2(12)$, it follows from the other Schwinger-Dyson relation Eq. (24) that

$$\frac{\delta I^{(2)}}{\delta G_2(12)}=\tfrac{1}{2}[i\sigma^{(2)}\partial_{t_1}\delta(1,2)+\gamma_3(123)G_1(3) + G_2^{-1}(12)+\Sigma(12;G_2)] . \tag{45}$$

Therefore, integrating with respect to G_2 gives

$$I^{(2)}(G_1,G_2,\gamma_3)=\tfrac{1}{2}\operatorname{tr}(i\sigma^{(2)}\partial_{t_1}G_2)+\tfrac{1}{2}\operatorname{tr}(\gamma_3 G_1 G_2) + \tfrac{1}{2}\operatorname{tr}(\ln G_2) + \tfrac{1}{2}\int_0^{G_2} dG_2'(12)\Sigma(12;G_2')+\Phi(G_1,\gamma_3) . \tag{46}$$

The G_2-independent term Φ is evaluated by a standard argument (see Sec. V B of [20]), with the result that up to a constant term it is the classical action with $\gamma_2=0$,

$$\Phi(G_1,\gamma_3)=S(G_1,\gamma_2=0,\gamma_3)+\Phi_0 , \tag{47}$$

where

$$S(G_1,\gamma_2,\gamma_3)=\tfrac{1}{2}G_1(1)i\sigma^{(2)}\partial_{t_1}G_1(1) + \frac{1}{2!}\gamma_2(12)G_1(1)G_1(2) + \frac{1}{3!}\gamma_3(123)G_1(1)G_1(2)G_1(3) . \tag{48}$$

The constant may also be found, but it is not important for our purposes. It is easy now to perform the inverse Legendre transform

$$I^{(1)}(G_1,\gamma_2,\gamma_3)=I^{(2)}(G_1,G_2,\gamma_3)+\frac{1}{2!}\gamma_2(12)M_2(12)\ , \tag{49}$$

in which G_2 is evaluated everywhere that it appears as the solution of the first Schwinger-Dyson equation Eq. (24). The result

$$\begin{aligned} I^{(1)}&(G_1,\gamma_2,\gamma_3)\\ &=S(G_1,\gamma_2,\gamma_3)-\tfrac{1}{2}\operatorname{tr}\ln[(G_2^{(0)})^{-1}-\gamma_3G_1-\Sigma(G_2)]\\ &\quad-\tfrac{1}{2}\operatorname{tr}(\Sigma G_2)+\tfrac{1}{2}\int_0^{G_2}dG_2'(12)\Sigma(12;G_2')+\text{const} \end{aligned} \tag{50}$$

is straightforwardly obtained, in which $G_2^{(0)}=(-i\sigma^{(2)}\partial_{t_1}1-\gamma_2)^{-1}$ is the bare propagator. It is helpful to introduce the two-point irreducible function $\Gamma_2=\gamma_3G_1+\Sigma$ and to use

$$\frac{\delta}{\delta G_2(12)}\operatorname{tr}(G_2\Sigma)=\Sigma(12)+\operatorname{tr}\left[G_2\frac{\delta\Sigma}{\delta G_2(12)}\right] \tag{51}$$

to combine the last two terms, with the final result

$$\begin{aligned} I^{(1)}(G_1,\gamma_2,\gamma_3)&=S(G_1,\gamma_2,\gamma_3)-\tfrac{1}{2}\operatorname{tr}\ln[1-G_2^{(0)}\Gamma_2(G_2)]\\ &\quad-\tfrac{1}{2}\int_0^{G_2}dG_2'(12)G_2'(34)\frac{\delta\Sigma(34;G_2')}{\delta G_2'(12)}\\ &\quad+\text{const}\ . \end{aligned} \tag{52}$$

This form of the exact $I^{(1)}$ is especially useful for approximation.

For example, within the approximation scheme developed above based on the $1/\sqrt{N}$ expansion the self-energy may be written as $\Sigma=\Sigma_{\rm DIA}+O(1/\sqrt{N})$, where $\Sigma_{\rm DIA}=\frac{1}{2}\gamma_3G_2G_2\gamma_3$. This latter is a quadratic polynomial in G_2, so that

$$G_2(34)\frac{\delta\Sigma_{\rm DIA}(34;G_2)}{\delta G_2(12)}=2\Sigma_{\rm DIA}(12;G_2)\ . \tag{53}$$

On the other hand,

$$\Sigma_{\rm DIA}(12;G_2)=2\frac{\delta\Omega_{\rm DIA}^{(2)}(G_2)}{\delta G_2(12)}\ , \tag{54}$$

where

$$\Omega_{\rm DIA}^{(2)}(G_2)=\frac{1}{2(3!)}\gamma_3G_2G_2G_2\gamma_3\ . \tag{55}$$

Hence the leading-order DIA-level approximation to the effective action is

$$\begin{aligned} I_{\rm DIA}^{(1)}&(G_1,\gamma_2,\gamma_3)\\ &=S(G_1,\gamma_2,\gamma_3)-\tfrac{1}{2}\operatorname{tr}\ln[1-G_2^{(0)}\Gamma_{2;\rm DIA}(G_2)]\\ &\quad-2\Omega_{\rm DIA}^{(2)}(G_2)+\text{const}\ . \end{aligned} \tag{56}$$

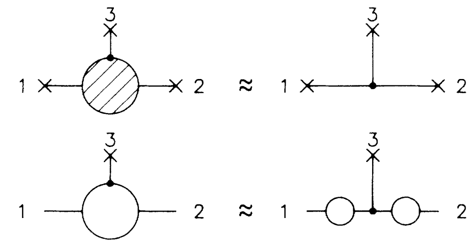

FIG. 10. Leading approximations to functional derivatives of two-point cumulant and irreducible functions.

These leading terms are all $\sim N$ and corrections are of lower order $O(\sqrt{N})$.

This effective action may be used to evaluate the irreducible functions Γ_p, calculated by pth-order functional derivatives of $I_{\rm DIA}^{(1)}$ with respect to G_1. Recalling the exact relations $\delta\Gamma_2/\delta G_1=\Gamma_3$ and $\delta G_2/\delta G_1=G_2\Gamma_3G_2$, we see that within the same level of approximation

$$\frac{\delta\Gamma_2(12)}{\delta G_1(3)}=\gamma_3(123) \tag{57}$$

and

$$\frac{\delta G_2(12)}{\delta G_1(3)}=\gamma_3(\overline{1}\overline{2}3)G_2(\overline{1}1)G_2(\overline{2}2)\ . \tag{58}$$

Graphically, these relations may be expressed as shown in Fig. 10. The corrections to these equations are of relative order $O(1/\sqrt{N})$. In general we shall denote equality of two expressions up to terms of such relative order $O(1/\sqrt{N})$ by $\approx$. Consider the term in the effective action

$$\begin{aligned} I_{\text{1-loop}}^{(1)}(G_1)&=-\tfrac{1}{2}\operatorname{tr}\ln[1-G_2^{(0)}\Gamma_{2;\rm DIA}(G_2)]\\ &=\sum_{n=1}^{\infty}\frac{1}{2n}\operatorname{tr}[(G_2^{(0)}\Gamma_{2;\rm DIA})^n]\ . \end{aligned} \tag{59}$$

Using Eq. (57), it follows that

$$\begin{aligned} \frac{\delta I_{\text{1-loop}}^{(1)}(G_1)}{\delta G_1(1)}&\approx\tfrac{1}{2}\gamma_3(123)\sum_{n=0}^{\infty}[G_2^{(0)}(\Gamma_{2;\rm DIA}G_2^{(0)})^n](23)\\ &=\tfrac{1}{2}\gamma_3(123)G_{2;\rm DIA}(23)\ . \end{aligned} \tag{60}$$

While the terms $\delta S/\delta G_1(1),\delta I_{\text{1-loop}}^{(1)}/\delta G_1(1)\sim\sqrt{N}$, note that

$$\begin{aligned} 2\frac{\delta\Omega_{\rm DIA}^{(2)}}{\delta G_1(1)}&\approx\tfrac{1}{2}\times(\text{graph in Fig. 11})\\ &=O(1)\ , \end{aligned} \tag{61}$$

so that it may be neglected at this level. In fact, we see from the above that retaining only

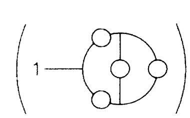

FIG. 11. Leading approximation to functional derivative of the irreducible 0-point function.

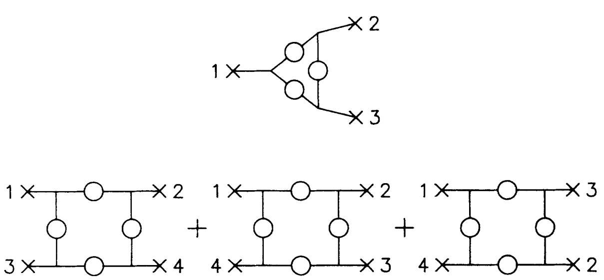

FIG. 12. One-loop contributions to the three- and four-point irreducible functions.

$-\gamma_1(1) \approx \delta S/\delta G_1(1) + \delta I^{(1)}_{1\text{-loop}}/\delta G_1(1)$, one obtains the exact relation $-\gamma_1 = -(G_2^{(0)})^{-1}G_1 + \frac{1}{2}\gamma_3 G_1^2 + \frac{1}{2}\gamma_3 G_2$ correctly up to terms of relative order $O(1/\sqrt{N})$. Now from the functional derivatives of $S(G_1)$ one obtains the bare contributions to the irreducible functions. Using Eq. (58), one can see further that successive functional derivatives of $I^{(1)}_{1\text{-loop}}$ with respect to G_1 simply insert one extra external line into a one-loop graph with bare vertices γ_3 and internal propagator lines given by $G_{2;\text{DIA}}$. Therefore, $\delta^p I^{(1)}_{1\text{-loop}}/\delta G_1(1) \cdots \delta G_1(p)$ is just the sum of all such one-loop p-point graphs. For example, the three- and four-point contributions are shown in Fig. 12. (Notice by the same argument that successive derivatives of the term we have neglected, $2\Omega^{(2)}_{\text{DIA}}$, generate a series of two-loop contributions.) Therefore, within the approximations we have made, the leading contribution to each irreducible function Γ_p, which is $\sim 1/\sqrt{N^{p-2}}$, is just the bare term plus the one-loop contributions with bare vertices γ_3 and internal propagator lines $G_{2;\text{DIA}}$.

We should stress again the nature of our approximations. We have expanded the effective action in terms of $1/\sqrt{N}$ and kept only the leading-order term $\sim N$. Thereafter, we evaluated functional derivatives, neglecting all terms of relative order $O(1/\sqrt{N})$. Although one might hope that these approximations should suffice to give all the leading-order contributions $\sim 1/\sqrt{N^{p-2}}$ to Γ_p, we have found no proof of this. Let us also note that at the DIA level of approximation as calculated above, all correlation functions have a finite-order expansion in terms of the bare vertex and DIA propagators. Therefore, the general arguments in [19] imply that for the modified Navier-Stokes system (MNS) or shell dynamics there will be no divergences whatsoever at this order. Any divergences which might be the signature of corrections to Kolmogorov scaling must appear—if at all—at higher orders in the expansion. We now give some discussion of this problem.

IV. "SPHERICAL SHELL MODELS" AND ANOMALOUS SCALING

A. Considerations on anomalous scaling

An important question is whether successive approximations in $1/\sqrt{N}$ beyond the DIA level, like those discussed in the preceding section, are adequate to calculate anomalous scaling exponents. If, for example, the anomalous exponents x_p in Eq. (20) had an asymptotic expansion of the form

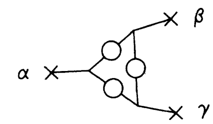

FIG. 13. One-loop vertex correction in the spherical model.

$$x_p = \sum_{k \geq 1} \frac{c_k^{(p)}}{N^{k/2}}, \tag{62}$$

then they should show up as logarithmic infrared divergences in the pth power velocity correlator evaluated at a given order in $1/\sqrt{N}$. Of course, as we have already discussed, it would be ill advised to attempt such an evaluation for the true Navier-Stokes dynamics, since the problems which already plague the Eulerian DIA will show up at every order. However, it would be reasonable to attempt such an evaluation for Kraichnan's MNS or for the shell models. So far we have made no serious attempt in the spherical versions of either of these models to analyze the divergence structure of the series of higher successive approximations generated above. It is actually not at all clear to us that any simple asymptotic expression like Eq. (62) should be valid for the models. In fact, the N dependence of velocity correlation functions in the models will certainly be much more complicated than simple power series in $1/\sqrt{N}$. Notice, for example, that one of the first terms contributing to the triple correlation $\langle v_\alpha v_\beta v_\gamma \rangle$ (with fixed $\alpha+\beta+\gamma=0$) in the subleading order $O(1/N)$ following the DIA term of order $O(1/\sqrt{N})$ is (Fig. 13) proportional to $\begin{pmatrix} J & J & J \\ -\alpha & \beta & \gamma \end{pmatrix} \times N^{3/2} \times \begin{Bmatrix} J & J & J \\ J & J & J \end{Bmatrix} = O(1/N)$. Even ignoring the Wigner $3j$ symbol, which is common to all terms and may be factored out, the expression contains the $6j$ symbol which is also rapidly oscillating as $N \to +\infty$. In fact, according to the Ponzano-Regge formula

$$\begin{Bmatrix} J & J & J \\ J & J & J \end{Bmatrix} \sim \left[\frac{32}{\pi^2}\right]^{1/4} N^{-3/2} \cos\left[3N\theta + \frac{\pi}{4}\right], \tag{63}$$

where θ is an irrational number defined by $\cos\theta = -\frac{1}{3}$. It seems that this type of rapidly oscillating behavior is the price one must pay for validity of the DIA in a deterministic model. However, it does not seem likely that such an oscillation, particularly with rapid changes of *sign*, could appear in the exponents x_p. (This would be very strange behavior in the shell model, since the difference there between relevant and irrelevant variables in the RG sense is just associated to the sign.) On the other hand, a behavior like

$$x_p \sim \frac{c^{(p)}}{N} \cos^2(3N\theta) + \cdots \tag{64}$$

would be odd, but not inconceivable. The variation of the x_p with N could also be such that it would not show up at all in the above approximations, which amount essentially to an expansion in $1/\sqrt{N}$. For example, if

$$x_p \sim c^{(p)} e^{-N}, \tag{65}$$

then the exponents have an essential singularity at the origin in the variable $1/\sqrt{N}$ and the previous approximations would yield a null result at every order. It is therefore a quite important question for judging the $1/\sqrt{N}$ expansion as an approximation technique to the exponents $x_p(N)$ to understand their variation in N.

However, the question has a significance beyond that of evaluating the $1/\sqrt{N}$ expansion idea. In fact, the SPM (or RCM) versions of the MNS or the shell dynamics are the only models we know in which the 1941 Kolmogorov theory (K41) demonstrably becomes exact in a limit. Considerable thought has been expended on this question without coming up with other clear examples of this sort. For instance, it has been speculated that K41 may become exact for Navier-Stokes dynamics in infinite dimension [22]. However, there is not even a clear heuristic argument that this should be true, let alone a proof. Therefore, the present models are the only "laboratory" presently available in which the deviations from K41 can be tuned to zero at will and in which the exact mechanisms restoring the K41 behavior can be studied. The spherical shell models are particularly convenient for this purpose because they can be studied by direct numerical simulation at high Reynolds number with relative ease. Therefore, we will say a few words about these models here.

B. Spherical shell models

A class of models which have recently been studied are the *complex shell models,* of a type represented by the Okhitani-Yamada model [9]. These models have the general structure

$$(\partial_t+\nu k_n^2)u_n(t)=i\sum_{m,l} A_{nml}u_m^*(t)u_l^*(t)+f_n(t)\ , \tag{66}$$

in which n ranges over an interval of integers $[-H,+K]$ and the wave number of the nth shell is $k_n=2^n k_0$. The interaction coefficient A_{nml} is real and

$$A_{nml}=k_n R_{m-n,l-n} \tag{67}$$

with $R_{p,q}=O(1)$ and "local," i.e., vanishing outside a finite range or even nearest neighbor. The models are also defined so that energy $E(t)=\frac{1}{2}\sum_n |u_n|^2$ is conserved. A Liouville theorem holds always since $\sum_n \mathrm{Re}\{\partial\dot{u}_n/\partial u_n\}=0$ is automatic. The term f_n is a force which is nonvanishing only in shells with $n\approx -H$, providing an input of energy at small wave number. It may be conveniently chosen to be Gaussian with a covariance

$$\langle f_n(t)f_m^*(t')\rangle=F_{nm}(t-t')\ . \tag{68}$$

An interesting feature of these models is that they possess a threefold "Potts symmetry" under the transformation

$$u_n'(t)=e^{2\pi i\tau/3}u_n(t),\quad \tau=0,1,2 \tag{69}$$

as an exact stochastic invariance (since the force may be redefined as $f_n'=e^{2\pi i\tau/3}f_n$ with the same distribution). Hence the steady state of this dynamics will be invariant under these Potts transformations, as will the time evolution of any initial distribution which is Z(3) invariant. This has the implication that the only nonvanishing two-point functions will be the covariance

$$U_{nm}(t,t')=\langle u_n(t)u_m^*(t')\rangle\ , \tag{70}$$

the response function

$$G_{nm}(t,t')=i\langle u_n(t)\hat{u}_m^*(t')\rangle=\left\langle\frac{\delta u_n(t)}{f_m(t')}\right\rangle\ , \tag{71}$$

and the "anti-response function"

$$\overline{G}_{nm}(t,t')=-i\langle \hat{u}_n(t)u_m^*(t')\rangle\ , \tag{72}$$

which obeys $\overline{G}_{nm}(t,t')=[G_{mn}(t',t)]^*$. (Notice that the three-point functions $\langle uuu\rangle$ and $\langle u^*u^*u^*\rangle$ can be nonvanishing and indeed give the mean energy transfer through the chain.) It is easy to write down shell DIA equations as approximate closure equations for the models. These have the explicit form

$$(\partial_t+\nu k_n^2)U_{nm}(t,t')=\sum_l\int_{t_0}^{t'}ds\,F_{nl}(t,s)\overline{G}_{lm}(s,t')+\frac{1}{2}\sum_{lk,pqr}A_{nlk}A_{rpq}\int_{t_0}^{t'}ds\,U_{lp}^*(t,s)U_{kq}^*(t,s)\overline{G}_{rm}(s,t')$$
$$+\sum_{lk,pqr}A_{nlk}A_{qpr}\int_{t_0}^{t}ds\,U_{lp}^*(t,s)G_{kq}^*(t,s)U_{rm}(s,t') \tag{73}$$

and

$$(\partial_t+\nu k_n^2)G_{nm}(t,t')=\delta_{n,m}\delta(t-t')+\sum_{lk,pqr}A_{nlk}A_{qpr}\int_{t'}^{t}ds\,U_{lp}^*(t,s)G_{kq}^*(t,s)G_{rm}(s,t')\ . \tag{74}$$

These equations may be directly verified to have some important properties, such as conservation of energy and existence of solutions obeying the fluctuation-dissipation relation for zero ν and f_n, as well as "integrability conditions" which guarantee that the results of integration are the same along any path in (t,t') space. Furthermore, a scaling analysis which is not reproduced here implies that with fixed input of energy $\bar{\epsilon}$ by the external force the equations have stationary solutions with K41 scaling.

By our preceding discussion these same equations arise as the exact solutions for $N\to+\infty$ of the following *spherical shell model* (SSM) equations:

$$(\partial_t+\nu k_n^2)u_{n\alpha}(t)$$
$$=\sum_{m\beta,l\gamma}A_{nml}w_N(\alpha,\beta,\gamma)u_{m\beta}^*(t)u_{l\gamma}^*(t)+f_{n\alpha}(t)\ , \tag{75}$$

with now

$$w_N(\alpha,\beta,\gamma)=\sqrt{N}\begin{bmatrix} J & J & J \\ \alpha & \beta & \gamma \end{bmatrix} . \tag{76}$$

As before, the variables α,β,γ range over integers from $-J$ to J and $u_{n\alpha}^*=(-1)^\alpha u_{n,-\alpha}$ is imposed. Hence there are $N=2J+1$ real modes per shell. Notice that this model does *not* coincide with the previous one for $N=1$: observe the missing factor of i and the fact that the imposed complex-conjugation symmetry implies that the zero modes u_{n0} are *real*. It is better to think of these SSM as a class of complex shell models in their own right which reduce at $N=1$ to a real-valued shell model:

$$(\partial_t+\nu k_n^2)u_n(t)=\sum_{m,l} A_{nml}u_m(t)u_l(t)+f_n(t) . \tag{77}$$

The important point here is that the previous DIA equations become rigorously exact for the SSM when $N\to+\infty$ (with the additional feature that the functions U and G appearing there are all real and the complex conjugations may be dropped). The previous observed consistency properties now all follow automatically from the existence of a model representation with the same properties. Furthermore, we see also that the two-point statistics of the SSM will obey K41 scaling in the large-N limit. In particular, the $\frac{5}{3}$ energy law will be observed.

Even stronger agreement with K41 appears in that limit. In fact, since $w_N=O(1/\sqrt{N})$ and each L-loop contribution to a pth-order cumulant G_p has $V=2L+p-2$ vertices, it follows that

$$G_p(\alpha_1 1,\ldots,\alpha_p p)=O\left[\frac{1}{\sqrt{N^{p-2}}}\right] , \tag{78}$$

with $\alpha_1,\ldots,\alpha_p$ a fixed set of modes, at every order of perturbation theory. The point here is that the L loop summations over initial indices exactly balance the $2L$ factors of $1/\sqrt{N}$ from the vertices [2]. Therefore, any fixed set of modes $\alpha_1,\ldots,\alpha_p$ should have *Gaussian statistics* in the limit $N\to+\infty$. In particular, for a fixed α in the SSM,

$$\langle |u_{n\alpha}|^{2p}\rangle_{(N)}\to C_p(\langle |u_n|^2\rangle_{\rm DIA})^p , \tag{79}$$

with $C_p=(2p)!/p!2^p$ the number of ways of pairing $2p$ objects. Note that the left-hand side of Eq. (79) is really independent of α by the SU(2) invariance of the steady state and the fact that the $u_{n\alpha}$ transform under an irreducible (therefore, cyclic) representation of SU(2). Equivalently, the "inertial-range flatnesses"

$$F_n^{(p)}(N)\equiv\frac{\langle |u_{n\alpha}|^{2p}\rangle_{(N)}}{(\langle |u_{n\alpha}|^2\rangle_{(N)})^p}\to C_p \tag{80}$$

and are independent of n, in the limit $N\to+\infty$. Therefore, the "intermittency exponents" $\xi_p(N)$ defined through

$$F_n^{(p)}(N)\sim k_n^{-\xi_p(N)} \tag{81}$$

should all vanish in the large-N limit.

An interesting problem which we have already raised—and which is probably feasible to study by direct numerical simulation—is the rate at which $\xi_p(N)\to 0$ for $N\to+\infty$. Another question of physical interest is the exact dynamical mechanism by which the K41 scaling is restored in the limit. As discussed by Kraichnan in Sec. 2 of [23], the K41 theory can only be consistent for Navier-Stokes equations if there is strong enough spatial diffusion of energy to suppress fluctuations of energy transfer on inertial-range scales. Similar considerations apply to the shell models. In fact, the strong deviations from Kolmogorov scaling there are observed to be accompanied by large temporal fluctuations in the energy transfer. A conspicuous feature in simulations of the shell dynamics is an intermittent "bursting" behavior of the energy flux [24,9]. Presumably these phenomena should disappear in the limit $N\to+\infty$. The mechanism must be very analogous to that considered by Kraichnan for the Navier-Stokes system, in which the sharp pulses of energy in the SPM are dispersed and smoothed out by complicated scattering and back-scattering among the very many, weakly interacting modes in adjacent shells. It would be worth verifying these theoretical considerations by direct numerical simulation of the SSM.

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